ICASE

A FINITE DIFFERENCE SCHEME FOR THE COMPRESSIBLE

NAVIER-STOKES EQUATIONS AND A RELATIONSHIP TO THE EULER EQUATIONS

Milton E. Rose

Report No. 81-21

July 22, 1981

(NASA-CR-185793) A FINITE DIFFERENCE SCHEME FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS AND A RELATIONSHIP TO THE EULER EQUATIONS (ICASE) 26 p

N89-71323

Unclas

00/34 0224378

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the

UNIVERSITIES SPACE USRA RESEARCH ASSOCIATION

A FINITE DIFFERENCE SCHEME FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS AND A RELATIONSHIP TO THE EULER EQUATIONS

Milton E. Rose
Institute for Computer Applications in Science and Engineering

ABSTRACT

This report adapts an earlier method described by Philips and Rose [3] to treat the compressible Navier-Stokes equations by an implicit system of compact finite difference equations. The boundary conditions for the related Euler problem are shown to follow formally from the finite difference equations as the viscosity vanishes by means of singular perturbation arguments.

This report was prepared as a result of work performed under NASA Contract No. NAS1-16394 and NAS1-15810 at ICASE, NASA Langley Research Center, Hampton, VA 23665.

1. Introduction

A recent paper (Philips and Rose [3]) described a compact finite difference scheme to treat the scalar convective-diffusion equation

$$u_t + a\,u_x + b\,u_y = c\,u_{xx} + 2\,d\,u_{xy} + e\,u_{yy}\,, \tag{1}$$

and indicated modifications required to treat systems of equations of this type when c, d, and e are nonsingular matrices. This paper describes the further modifications which are necessary to extend this treatment to the Navier-Stokes equations, a case in which c, d, and e are each singular. The result shows a close, but not self-evident, correspondence to the scheme for nonsingular coefficient matrices. The extended scheme and the modified development which leads to it are described in the next section.

The Euler equations arise as the formal singular perturbation limit ($\mu \to 0$, μ = shear coefficient of viscosity) of the Navier-Stokes equation and it may be conjectured that the physically relevant generalized solutions of the Euler equations as well as the associated class of correct mathematical boundary conditions for them are determined as the "outer expansions" of the Navier-Stokes equations in the sense of singular perturbation theory.

A simple energy argument suggests that the Navier-Stokes equations are well-posed under boundary conditions which are independent of the Mach number (Problem P). In contrast, the theory of characteristics for hyperbolic equations shows that the number of boundary conditions for the Euler equations depends upon the Mach number and is, generally, less than the number of boundary conditions which are appropriate for the Navier-Stokes equations. This reduction in the number of boundary conditions is a characteristic feature of singular perturbation problems.

A study of the finite-difference scheme (3.9) for the Navier-Stokes equations shows that only the Euler boundary conditions have an appreciable effect on the solution of (3.9) as $\mu \to 0$. As a result the difference scheme (3.9) provides for a treatment of both problems under boundary conditions which are independent of the Mach number.

2. The Navier-Stokes Equations

If ρ, $\underline{u} = (u,v)$, T are the density, velocity, and temperature, respectively and if

$$\pi' = \begin{pmatrix} 2\mu u_x + \lambda \operatorname{div} \underline{u} & \mu(u_x + v_y) \\ \mu(u_x + v_y) & 2\mu v_y + \lambda \operatorname{div} \underline{u} \end{pmatrix},$$

is the reduced stress tensor, the Navier-Stokes equations in two-dimensions may be written

$$\begin{aligned} &\rho_t + (\underline{u} \cdot \nabla)\rho + \rho \operatorname{div} \underline{u} = 0 \\ (2.1) \qquad &\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} + \rho^{-1} R T \operatorname{grad} \rho + R \operatorname{grad} T = \rho^{-1} \operatorname{div} \pi' \\ &T_t + (\underline{u} \cdot \nabla)T + (\gamma-1)T \operatorname{div} \underline{u} = (\rho c_v)^{-1} k \operatorname{div} \operatorname{grad} T + f, \end{aligned}$$

where

$$f = (\rho c_v)^{-1} \mu[u_x^2 + v_y^2 + 2(u_y + v_x)^2 + \lambda(\operatorname{div} \underline{u})^2].$$

Here k, μ, λ, c_v, c_p are the coefficient of heat conduction, the shear and second coefficients of viscosity, and the specific heats at constant volume and pressure, respectively; also, $R = c_p - c_v$ and $\gamma = c_p/c_v$.

Equations (2.1) may be expressed in the form

$$(2.2) \qquad U_t + AU_x + BU_y = CU_{xx} + 2DU_{xy} + EU_{yy} + F,$$

where the transpose of the vector U is given by (ρ,u,v,T) and

$$A = \left(\begin{array}{c|ccc} u & \rho & 0 & 0 \\ \hline RT/\rho & u & 0 & R \\ 0 & \hat{0} & u & 0 \\ 0 & (\gamma-1)T & 0 & u \end{array}\right) = \left(\begin{array}{c|c} a_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

$$B = \left(\begin{array}{c|ccc} v & 0 & \rho & 0 \\ \hline 0 & v & 0 & 0 \\ RT/\rho & 0 & v & R \\ 0 & 0 & (\gamma-1)T & v \end{array}\right) = \left(\begin{array}{c|c} b_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$C = \mu\rho^{-1} \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 2+\lambda/\mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma/P_r \end{array}\right) = \mu\rho^{-1} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & C_{22} \end{array}\right)$$

(2.3)

$$D = \frac{\mu\rho^{-1}}{2} \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1+\lambda/\mu & 0 \\ 0 & 1+\lambda/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E = \mu\rho^{-1} \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 2+\lambda/\mu & 0 \\ 0 & 0 & 0 & \gamma/P_r \end{array}\right) = \mu\rho^{-1} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{22} \end{array}\right)$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix} ,$$

in which $P_r = \frac{\mu c_p}{k}$ is the Prandtl number.

It will be convenient to introduce the matrices

$$\begin{aligned} J &= \operatorname{diag}(0,1,1,1), \\ \hat{J} &= I - J = \operatorname{diag}(1,0,0,0). \end{aligned} \tag{2.4}$$

With this definition equations (2.2) may be written in system form as

$$\begin{aligned} U_t + A U_x + B U_y &= J(V_x + W_y + F) \\ C U_x + D U_y &= J V \\ D U_x + E U_y &= J W . \end{aligned} \tag{2.5}$$

Because the elliptic operator on the right-hand side of equation (2.2) has rank 3, it is not immediately apparent how boundary conditions may be imposed. To this end, consider the one-dimensional problem

$$U_t + A U_x = C U_{xx}, \tag{2.6}$$

where A is symmetric,

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} ,$$

and $a_{11} > 0$ while

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_{22} \end{pmatrix} ,$$

is non-negative. With initial and boundary conditions given by

$$(2.7)\qquad \begin{aligned} &\text{a)}\quad U(x,0) = \overline{U}\ , \\ &\text{b)}\quad U(0,t) = 0\ , \\ &\text{c)}\quad JU(1,t) = 0\ , \end{aligned}$$

if (2.6) is multiplied by U^T and then integrated the result is the "energy" expression

$$0 = \tfrac{1}{2}\frac{d}{dt}\int_0^1 U^T U dx + \int_0^1 U_x^T C U_x + U^T(\tfrac{1}{2}AU - CU_x)\Big|_0^1 .$$

Employing the initial and boundary conditions (2.7) and noting that $a_{11} > 0$ by assumption, there results

$$(2.8)\qquad \int_0^1 U^T(x,t)U(x,t)dx \le \int_0^1 \overline{U}^T(x)\overline{U}(x)dx,$$

where the equality applies if and only if U_x = const. This, of course, implies the uniqueness of the solution for the linear problem considered.

Applied to the hydrodynamic problem (2.2), $a_{11} = u$ in which case the boundary conditions (2.7b) and (2.7c) correspond to inflow and outflow conditions. We thus state:

Problem P: Solve the Navier-Stokes equations in the form (2.5) in a domain $\mathcal{D}$ under the initial and boundary conditions

$$(2.9)\qquad \begin{aligned} U(x,y,0) &= \overline{U} \\ U(\cdot,t) &= U \text{ inflow} \\ JU(\cdot,t) &= JU \text{ outflow.} \end{aligned}$$

More specifically, we assume $\mathcal{D}$ is the unit square on which inflow conditions apply for $x=0$ or $y=0$ while outflow conditions apply for $x=1$ or $y=1$.

A more complete discussion of properly-posed boundary conditions for problems of the type considered in this paper has been given by Strikwerda [4].

3. A Compact Finite Difference Scheme

If we ignore for the time being the fact that the term F in (2.4) is a function of U_x and U_y this equation is similar to the type of problem which was treated earlier by means of a second-order accurate compact finite difference scheme (Philips and Rose [3]). However, their argument depended essentially upon the fact that the coefficient matrices C, D, E in (2.4) were nonsingular; in order to describe the extension necessary when these coefficients are singular (compare (2.3)) it appears simplest to rederive the derivation of the difference equations from elementary principles. This is done here.

We suppose the computational domain can be subdivided into rectangular computational cells $\pi^n_{jk}\{|x - x_j| < \Delta x/2,\ |y - y_k| < \Delta y/2,\ |t - t_n| < \frac{\Delta t}{2}\}$ and write $U^n_{jk} = U(j\,\Delta\,x, k\,\Delta\,y, n\,\Delta\,t)$. We employ the notation

$$\mu_x U^n_{jk} = (U^n_{j+\frac{1}{2},k} + U^n_{j-\frac{1}{2},k})/2 \quad ,$$

$$\delta_x U^n_{jk} = (U^n_{j+\frac{1}{2},k} - U^n_{j-\frac{1}{2},k})/\Delta x \quad , \tag{3.1}$$

etc. When no confusion is likely to arise we suppress the spatial indices by writing $U^n_{\bullet} = U(\cdot,\cdot,n\,\Delta\,t)$; thus $\mu_x U^n_{\bullet}$, $\delta_x U^n_{\bullet}$, $\mu_y U^n_{\bullet}$, $\delta_y U^n_{\bullet}$, $\mu_t U^n_{\bullet}$, $\delta_t U^n_{\bullet}$ involve the values of U at the center points of the faces of the cell $\pi^n_{\bullet}$. A finite difference scheme which only involves data associated with $\pi^n_{\bullet}$ is called compact.

The approximation method to be described is based upon the following idea: suppose the solution U of (2.4) is known to be smooth; then the result of approximating the coefficient matrices in (2.4) by their values averaged over each computational cell π^n, say $A^n_\bullet$, $B^n_\bullet$, etc., leads to a linear partial differential equation in each cell.

$$
\text{(3.2)} \qquad
\begin{aligned}
U_t + A^n_\bullet U_x + B^n_\bullet U_y &= J(V_x + W_y + F^n_\bullet),\\
C^n_\bullet U_x + D^n_\bullet U_y &= J\,V_x,\\
D^n_\bullet U_x + E^n_\bullet U_y &= J\,W_x.
\end{aligned}
$$

This system will approximate (2.4) to terms of second-order in the mesh parameters if π^n is sufficiently small. Because (3.2) is linear it is feasible to construct a linear manifold of solutions in each cell and then, by means of algebraic equations which express continuity conditions at the boundaries of neighboring cells together with the initial and boundary conditions associated with the problem, determine the parameters which lead to an approximation to the solution of (2.4). These algebraic conditions are expressed by the finite difference equations (3.9) whose development we now describe.

The following discussion concerns (3.2) in a fixed cell $\pi^n_\bullet$; we therefore omit the indices $\bullet$, n on the coefficients in this systems of equations. With the coefficient matrices partitioned as in (2.3) introduce the following definitions: I_3 is the 3×3 identity matrix and

$$\tilde{A} \equiv \begin{pmatrix} 0 & -a_{11}^{-1}A_{12} \\ 0 & I_3 \end{pmatrix} , \qquad \tilde{B} \equiv \begin{pmatrix} 0 & -b_{11}^{-1}B_{12} \\ 0 & I_3 \end{pmatrix} ,$$

$$\tilde{A}_{22} \equiv A_{22} - A_{21}a_{11}^{-1}A_{12} , \qquad \tilde{B}_{22} \equiv B_{22} - B_{21}b_{11}^{-1}B_{12} ,$$

$$\omega_x \equiv C_{22}^{-1}\ \tilde{A}_{22} , \qquad \theta_x \equiv \frac{\Delta x}{2}\,\omega_x ,$$

(3.3)
$$\omega_y \equiv E_{22}^{-1}\ \tilde{B}_{22} , \qquad \theta_y \equiv \frac{\Delta x}{2}\,\omega_y ,$$

$$\Omega(\omega x) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \exp\ \omega x \end{pmatrix} ,$$

$$[AJ]^{-1} \equiv \begin{pmatrix} \sigma & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \qquad (\sigma \text{ arbitrary}).$$

We shall need to employ several results for which this notation will prove useful: First, the algebraic system of equations $AJX = Y$ may be verified to have the solution $X = [AJ]^{-1}Y$. Second, the system of differential equations $AY = JCY'$ has the general solution $Y(x) = \tilde{A}\Omega(\omega_x \cdot x)\underline{\alpha}$ where $\underline{\alpha}$ is a vector parameter.

Each of the terms I, $(xI - tA)$, $(yI - tB)$, $\tilde{A}\Omega(\omega_x \cdot x)$, $\tilde{B}\Omega(\omega_y \cdot y)$ is thus a solution of (3.2) so that

(3.4)
$$U = \underline{\alpha}_1 + (xI - tA)\underline{\alpha}_2 + (yI - tB)\underline{\alpha}_3 + \tilde{A}\Omega(\omega_x \cdot x)\underline{\alpha}_4 + \tilde{B}\Omega(\omega_y \cdot y)\underline{\alpha}_5 + tJF,$$

describes a solution manifold of (3.2). Set

$$JV = CU_x + DU_y,$$

$$JW = DU_x + EU_y.$$

The ten values of the vectors $U_{\bullet}^{n\pm\frac{1}{2}}$, $U_{j\pm\frac{1}{2},k}^{n}$, $U_{j,k\pm\frac{1}{2}}^{n}$, $V_{j\pm\frac{1}{2},k}^{n}$, $W_{j,k\pm\frac{1}{2}}^{n}$ may be assumed to be continuous across contiguous cells. The partial result of expressing the parameters $\{\underline{\alpha}_i\}$ in terms of these values is

$$(\delta_t + A\delta_x + B\delta_y)U_{\bullet}^n = A\tilde{A}\delta_x\Omega(\omega_x \cdot x)\underline{\alpha}_4 + B\tilde{B}\delta_y\Omega(\omega_y \cdot y)\underline{\alpha}_5 + JF$$

$$\mu_t U_{\bullet}^n = \mu_x U_{\bullet}^n + 0(\Delta x^2) = \mu_y U_{\bullet}^n + 0(\Delta y^2)$$

(3.5)
$$(C\delta_x + D\delta_y)U_{\bullet}^n = J\mu_x U_{\bullet}^n - C\tilde{A}\left(\mu_x \frac{d}{dx}\Omega(\omega_x \cdot x) - \delta_x\Omega(\omega_x \cdot x)\right)\underline{\alpha}_4$$
$$- D\tilde{B}\left(\mu_x \frac{d}{dx}\Omega(\omega_y \cdot y) - \delta_y\Omega(\omega_y \cdot y)\right)\underline{\alpha}_5$$

$$(D\delta_x + E\delta_y)U_{\bullet}^n = J\mu_y \omega_{\bullet}^n - E\tilde{B}\left(\mu_y \frac{d}{dy}\Omega(\omega_y \cdot y) - \delta_y\Omega(\omega_y \cdot y)\right)\underline{\alpha}_5$$
$$- D\tilde{B}\left(\mu_y \frac{d}{dx}\Omega(\omega_x \cdot x) - \delta_x\Omega(\omega_x \cdot x)\right)\underline{\alpha}_4 ,$$

where $\underline{\alpha}_4$ and $\underline{\alpha}_5$ are to be determined by solving

(3.6)
$$J\delta_y V_{\bullet}^n = C\tilde{A}\delta_x \cdot \frac{d}{dx}\Omega(\omega_x \cdot x) \cdot \underline{\alpha}_4$$

$$J\delta_y W_{\bullet}^n = E\tilde{B}\delta_y \cdot \frac{d}{dy}\Omega(\omega_y \cdot y) \cdot \underline{\alpha}_5 .$$

The solution of the latter pair of equations is

(3.7)
$$\underline{\alpha}_4 = [C\tilde{A}\delta_x \cdot \frac{d}{dx}\Omega(\omega_x \cdot x)]^{-1} \cdot J\delta_x V_{\bullet}^n ,$$

$$\underline{\alpha}_5 = [E\tilde{B}\delta_y \cdot \frac{d}{dy}\Omega(\omega_y \cdot y)]^{-1} \cdot J\delta_y W_{\bullet}^n ,$$

using the notation in (3.3) and noting that $C\tilde{A} = CJ$, $E\tilde{B} = EJ$.

Introduce the definitions

$$
(3.8)\qquad
\begin{aligned}
q(\theta) &\overset{\text{def}}{\equiv} \coth\theta - \theta^{-1}, \\
r(\theta) &\overset{\text{def}}{\equiv} 1 - \theta(\sinh\theta)^{-1}.
\end{aligned}
$$

The result of employing (3.7) in (3.6) is, finally,

$$
(3.9)\qquad
\begin{aligned}
&\text{a)}\quad (\delta_t + A\delta_x + B\delta_y)U^n_{\cdot} = J(\delta_x V^n_{\cdot} + \delta_y W^n_{\cdot} + F^n_{\cdot}) \\
&\text{b)}\quad \mu_t U^n_{\cdot} = \mu_x U^n_{\cdot} = \mu_y U^n_{\cdot} \\
&\text{c)}\quad (C\delta_x + D\delta_y)U^n_{\cdot} = (J\mu_x - \tfrac{\Delta x}{2} Q_x \cdot \delta_x)V^n_{\cdot} + R_y\delta_y W^n_{\cdot} \\
&\text{d)}\quad (D\delta_x + E\delta_y)U^n_{\cdot} = (J\mu_y - \tfrac{\Delta y}{2} Q_y \cdot \delta_y)W^n_{\cdot} + R_x\delta_x V^n_{\cdot}\ ,
\end{aligned}
$$

in which

$$
(3.10)\qquad
\begin{aligned}
Q_x &\overset{\text{def}}{\equiv} \begin{pmatrix} 0 & 0 \\ 0 & C_{22}q(\theta_x)C_{22}^{-1} \end{pmatrix}, \\
Q_y &\overset{\text{def}}{\equiv} \begin{pmatrix} 0 & 0 \\ 0 & E_{22}q(\theta_y)E_{22}^{-1} \end{pmatrix}, \\
R_x &\overset{\text{def}}{\equiv} \begin{pmatrix} 0 & 0 \\ 0 & D_{22}r(\theta_x)\tilde{A}_{22}^{-1} \end{pmatrix}, \\
R_y &\overset{\text{def}}{\equiv} \begin{pmatrix} 0 & 0 \\ 0 & D_{22}r(\theta_y)\tilde{B}_{22}^{-1} \end{pmatrix},
\end{aligned}
$$

where θ_x, θ_y are defined by (3.3).

For real values of θ the functions $q(\theta)$ and $r(\theta)$ given by (3.8) are regular in θ and are conveniently evaluated by

(3.11)

$$\text{(a)} \quad \begin{aligned} q(\theta) &\simeq \theta/3 \quad , \quad \theta \text{ small} \\ &\simeq \operatorname{sgn}\theta \, , \quad \theta \text{ large} \end{aligned}$$

where $\operatorname{sgn}\theta = \theta/|\theta|$; also

$$\text{(b)} \quad \begin{aligned} r(\theta) &\simeq \theta^2/6 \quad , \quad \theta \text{ small} \\ &\simeq 1 \quad , \quad \theta \text{ large.} \end{aligned}$$

The matrices θ_x, θ_y given by (3.3) are generalizations of the cell Reynolds number. Consider θ_x: if S is the matrix which diagonalizes θ_x, say $S^{-1}\theta_x S = \tilde{\theta}_x$, then

$$\begin{aligned} q(\theta_x) &= S\, q(\tilde{\theta}_x) S^{-1}, \\ r(\theta_x) &= S\, r(\tilde{\theta}_x) S^{-1}, \end{aligned} \tag{3.12}$$

and the approximations given in (3.11) may be used to evaluate $q(\tilde{\theta}_x)$, $r(\tilde{\theta}_x)$.

As mentioned earlier, the difference equations (3.9) generalize similar equations which were described in Philips and Rose [3] when the matrices C, D, and E were nonsingular. Arguments given there may be used to show that the truncation error in (3.9) is second order in the mesh parameters independent of θ_x, θ_y.

The reader is asked to verify the fact that the algebraic equations expressed by (3.9) together with (2.9) lead to a determined system of equations for $U^n_{\bullet}$, $V^n_{\bullet}$, and $W^n_{\bullet}$. When the coefficient matrices in (3.9) are symmetric and constant an energy-norm estimate for the solution may be given (cf. [3]); this shows the existence and uniqueness of the solution and also implies the convergence of the scheme any fixed values of the mesh parameters

$\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$. It is plausible that similar results hold when the coefficient matrices in (3.9) are variable and we appeal to this plausibility argument in the following discussion without explicit comment.

4. Solution Methods

a) As described in [3], compact schemes of the type (3.9) may be solved by a

two-step method:

i) by eliminating the value $U_\bullet^{n+\frac{1}{2}}$ common to (3.9a) and (3.9b) there results, with $\tau = \Delta t/2$,

$$(4.1)\qquad \begin{aligned} P_x \begin{pmatrix} U_\bullet^n \\ V_\bullet^n \end{pmatrix} + \tau R_y \begin{pmatrix} U_\bullet^n \\ W_\bullet^n \end{pmatrix} &= \begin{pmatrix} U_\bullet^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F_\bullet^n \quad , \\ P_y \begin{pmatrix} U_\bullet^n \\ W_\bullet^n \end{pmatrix} + \tau R_x \begin{pmatrix} U_\bullet^n \\ V_\bullet^n \end{pmatrix} &= \begin{pmatrix} U_\bullet^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F_\bullet^n \end{aligned}$$

where

$$P_x \overset{\text{def}}{\equiv} \begin{pmatrix} \mu_x + \tau A\delta_x & -\tau\delta_x J \\ C\delta_x & \frac{\Delta x}{2} Q_x\delta_x - \mu_x J \end{pmatrix} \quad ,$$

$$(4.2)\qquad P_y \overset{\text{def}}{\equiv} \begin{pmatrix} \mu_y + \tau B\delta_y & -\tau\delta_y J \\ E\delta_y & \frac{\Delta x}{2} Q_y\delta_y - \mu_y J \end{pmatrix} \quad ,$$

$$R_x \overset{\text{def}}{\equiv} \begin{pmatrix} A\delta_x & -\delta_x J \\ \tau^{-1} D\delta_x & -\tau^{-1} R(\theta_x)\delta_x \end{pmatrix} \quad ,$$

$$\mathcal{R}_y \overset{\text{def}}{\equiv} \begin{pmatrix} B\delta_y & -\delta_y J \\ \tau^{-1} D\delta_y & -\tau^{-1} R(\theta_y)\delta_y \end{pmatrix} .$$

The solution of (4.2) is determined by $U_\bullet^{n-\frac{1}{2}}$ and the imposed boundary conditions for $U_\bullet^n$. A formal ADI solution of (4.1), accurate to $O(\tau^2)$, is given by

(4.3)

$$\begin{pmatrix} U_\bullet^n \\ V_\bullet^n \end{pmatrix} = \mathcal{P}_x^{-1}(I - \tau\mathcal{R}_y\mathcal{P}_y^{-1}) \left[\begin{pmatrix} U_\bullet^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F_\bullet^n \right] ,$$

$$\begin{pmatrix} U_\bullet^n \\ W_\bullet^n \end{pmatrix} = \mathcal{P}_y^{-1}(I - \tau\mathcal{R}_x\mathcal{P}_x^{-1}) \left[\begin{pmatrix} U_\bullet^{n-\frac{1}{2}} \\ 0 \end{pmatrix} + \tau F_\bullet^n \right] .$$

(ii) Using the solution $U_\bullet^n$, $V_\bullet^n$, $W_\bullet^n$ obtained from (4.1) $U_\bullet^{n+\frac{1}{2}}$ may be calculated from either the "leapfrog" equation (3.9a) or from (3.9b)

In employing (4.1) the coefficient matrices are assumed to be evaluated at the center point of the cell π^n. We shall not pause to indicate how this may be approximated.

A drawback in employing (4.3) to solve (4.1) is that Δt must be suitably restricted; when the viscosity μ in (2.3) is sufficiently small this restriction is approximated by the CFL condition for the dominant hyperbolic part of the operator in (2.2). Presumably, in view of earlier remarks, (4.1) is solvable for any value of the ratio of mesh parameters λ_x, λ_y. In order to exploit this, particularly for the calculation of steady-state solutions of (3.9), a more effective solution method than (4.3) is required. This topic will not be treated here, however.

We remark, finally, that the existence of the unique solution of the algebraic equations (4.1) is a consequence of the (assumed) existence and uniqueness of the finite difference equations (3.9).

b) The operators $\mathcal{P}_x, \mathcal{P}_y$ in (4.3) involve the solution of algebraic two-point boundary value problems which can be obtained by a method due to Keller [1]. A simpler solution method results by observing that $U^n_{\bullet}$ may be directly obtained by solving a block tridiagonal system of equations (cf. [3]) as will now be shown. The asymptotic consequences when $\mu \to 0$ will be described in section 6.

The solution of

$$\mathcal{P}_x \begin{pmatrix} U^n_j \\ V^n_j \end{pmatrix} = \begin{pmatrix} g_{1,j} \\ g_{2,j} \end{pmatrix} ,$$

typifies the problem involved in applying (4.3) where $\mathcal{P}_x$ is given by (4.2). In a cell π^n_j these equations can be solved for the values $V^n_{j\pm\frac{1}{2}}$ with the result

$$\text{(4.5)} \qquad \begin{aligned} \lambda_x V^n_{j+\frac{1}{2}} &= [J]^{-1}(a^+_j\, U^n_{j+\frac{1}{2}} + b^+_j\, U^n_{j-\frac{1}{2}} - g^+_j) , \\ \lambda_x V^n_{j-\frac{1}{2}} &= [J]^{-1}(a^-_j\, U^n_{j+\frac{1}{2}} + b^-_j\, U^n_{j-\frac{1}{2}} + g^-_j) , \end{aligned}$$

in which

$$\text{(4.6)} \qquad \begin{aligned} \lambda_x &= \Delta t/\Delta x, \qquad \kappa_x = 2\lambda x/\Delta x , \\ a^{\pm} &= \tfrac{1}{2}[(Q_x \pm I)(I + \lambda_x A) + \kappa_x C], \\ b^{\pm} &= \tfrac{1}{2}[(Q_x \pm I)(I - \lambda_x A) - \kappa_x C], \end{aligned}$$

and

$$g^{\pm} = [(I \pm Q_x)g_1 \pm \lambda_x g_2].$$

The pair of values $U^n_{j+\frac{1}{2}}$, $V^n_{j+\frac{1}{2}}$ are common to the contiguous cells π^n_j, π^n_{j+1}. Expressions for the value $V^n_{j+\frac{1}{2}}$ in each such cell are given by (4.5); the result of equating these expressions for $V^n_{j+\frac{1}{2}}$ and setting $\ell = j+\frac{1}{2}$ is

$$(4.7) \qquad -Ja^-_{\ell+\frac{1}{2}} U^n_{\ell+1} + b^+_{\ell-\frac{1}{2}} U^n_{\ell-1} + (a^+_{\ell-\frac{1}{2}} - Jb^-_{\ell+\frac{1}{2}})U^n_\ell = g^+_{\ell-\frac{1}{2}} + Jg^-_{\ell+\frac{1}{2}} \,.$$

This block-tridiagonal system of equations may be efficiently solved for $U^n_\bullet$ with the boundary conditions prescribed by (2.9) and the values $V^n_\bullet$ can then be obtained from (4.5). However, in order to evaluate the coefficient matrices $a^\pm$, $b^\pm$ an effective means of approximating the matrix Q_x defined by (3.10) must be considered. This topic, with specific reference to the Navier-Stokes equations (2.2) is the subject of the next section.

5. The Matrices Q_x, Q_y, R_x, R_y

The matrix Q_x occurring in the coefficients a ,b in (4.6) was defined in terms of A and C by (3.10) in terms of the matrix $q(\theta_x)$, which itself was defined by (3.3) and (3.8). The matrix Q_y is similarly defined in terms of the matrices B and E.

Confining our attention to Q_x, first note that Q_x is given, using (3.3), by

$$(5.1) \qquad \theta_x = \frac{\rho\Delta x}{2\mu}\begin{pmatrix} (u - RT/u)\delta & 0 & R\delta \\ 0 & u & 0 \\ \varepsilon(\gamma-1)T & 0 & \varepsilon u \end{pmatrix} \,,$$

in which

$$\delta = (2+\lambda/\mu)^{-1}, \qquad \varepsilon = P_r/\gamma \quad . \tag{5.2}$$

Denote the eigenvalues of $(\frac{2\mu}{\rho\Delta x})\theta_x$ by $\theta_{x,1}, \theta_{x,2}, \theta_{x,3}$. If

$$\sigma = \varepsilon/\delta, \tag{5.3}$$

then

$$\theta_{x,1} = u, \tag{5.4}$$

and $\theta_{x,2}, \theta_{x,3}$ are given as

$$\frac{2\theta_{x,2,3}}{u\delta} = \left(1+\sigma - \frac{1}{\gamma M^2}\right) \pm M^{-1}\left[M^2\left((1+\sigma) - \frac{1}{\gamma M^2}\right)^2 + 4\sigma(1-M^2)\right]^{\frac{1}{2}}, \tag{5.5}$$

where $M = u/c$, $c^2 = \gamma RT$. The following approximations result:

M = 1

$$\frac{2\theta_{x,2}}{u\delta} = 2(1+\sigma - \frac{1}{\gamma}) > 0, \qquad \frac{2\theta_{x,3}}{u\delta} = 0 \; . \tag{5.6}$$

M small

$$\frac{2\theta_{x,2}}{u\delta} \simeq 1+\sigma > 0, \qquad \frac{2\theta_{x,3}}{u\delta} \simeq (1+\sigma) - \frac{2}{\gamma M^2} < 0. \tag{5.7}$$

M large

$$\frac{\theta_{x,2}}{u\delta} \simeq 1, \qquad \frac{\theta_{x,3}}{u\delta} \simeq \sigma. \tag{5.8}$$

Writing

$$\hat{\theta}_x = \begin{pmatrix} \theta_{x,1} & 0 & 0 \\ 0 & \theta_{x,2} & 0 \\ 0 & 0 & \theta_{x,3} \end{pmatrix} , \tag{5.9}$$

then

$$\theta_x = \frac{\rho\Delta x}{2\mu} S \cdot \hat{\theta}_x \cdot S^{-1} , \tag{5.10}$$

where

$$S = \begin{pmatrix} 0 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} ,$$

(5.11)

$$S^{-1} = \begin{pmatrix} 0 & s_3 - s_2 & 0 \\ -1 & 0 & s_3 \\ 1 & 0 & -s_2 \end{pmatrix} \div (s_3 - s_2) ,$$

in which

$$s_\nu = \frac{(\theta_{x,\nu} - \varepsilon u)}{\varepsilon(\gamma - 1)T} , \qquad \nu = 2,3. \tag{5.12}$$

As a result, using (3.12)

$$q(\theta_x) = S\, q\left(\left(\frac{\rho\Delta x}{2\mu}\right) \hat{\theta}_x\right) S^{-1} , \tag{5.13}$$

in which $q\left(\left(\frac{\rho\Delta x}{2\mu}\right)\hat{\theta}_x\right)$ may be approximated by using (3.8).

In view of (5.6), (5.7), (5.8),

(5.14)
$$\lim_{\mu \to 0} q(\theta_x) = S\left(\operatorname{sgn} u \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_x \end{pmatrix}\right) S^{-1} ,$$

where

$$\begin{aligned} \xi_x &= -1 , & M_x &< 1 \\ &= 0 , & M_x &= 0 \\ &= 1 , & M_x &> 1. \end{aligned}$$

where $M_x = u/c$.

Thus,

(5.16)
$$Q_x = \begin{pmatrix} 0 & 0 \\ 0 & c_{22} S\left[\operatorname{diag}\left(q(\theta_{x,1}), q(\theta_{x,2}), q(\theta_{x,3})\right)\right] S^{-1} c_{22}^{-1} \end{pmatrix} ,$$

and

(5.17)
$$\lim_{\mu \to 0} Q_x = \begin{pmatrix} 0 & 0 \\ 0 & c_{22} S\left(\operatorname{diag}(1,1,\xi_x)\right) S^{-1} \operatorname{sgn} u \cdot c_{22}^{-1} \end{pmatrix} .$$

Similar expressions result for Q_y noting (3.10).

In the same manner, using (3.8) and (3.10),

(5.18)
$$R_x = \begin{pmatrix} 0 & 0 \\ 0 & d_{22} S\left[\operatorname{diag}\left(r(\theta_{x,1}) r(\theta_{x,2}) r(\theta_{x,3})\right)\right] S^{-1} \tilde{a}_{22}^{-1} \end{pmatrix} ,$$

$$\lim_{\mu\to 0} R_x = \begin{pmatrix} 0 & \\ 0 & d_{22} S \,\mathrm{diag}(1,1,1)\, S^{-1}\tilde{a}_{22} \end{pmatrix} \tag{5.19}$$

with similar results for R_y.

Using (3.8) and (3.11), the results of this section allow the coefficient matrices Q and R in (3.9) to be evaluated as well as the coefficient matrices in (4.7) as described by (4.6).

6. The Euler Equations

The Euler equations

$$U_t + A\,U_x + B\,U_y = 0, \tag{6.1}$$

arise as the formal limit of the Navier-Stokes equations(2.5) as the viscosity $\mu \to 0$. If $U(\mu)$ denotes the solution of the Navier-Stokes equations with certain initial and boundary conditions, singular perturbation methods provide an important means of describing the sense in which $U(\mu)$ may be approximated by a solution U of the Euler equations (6.1) in regions exterior to boundary layers, shocks, etc. where vorticity can be generated.

The solution $U^n_{\cdot}(\mu)$ of the finite difference equations (3.9) together with (3.4) determines an approximate solution, say $U(\mu,\Delta x)$, of $U(\mu)$ if we assume that $U(\mu,\Delta x) \to U(\mu)$ as $\Delta x \to 0$. The construction employed in (3.4) is similar in viewpoint to one which could be employed by a singular perturbation method if one were to allow a much greater degree of algebraic complexity to be used in order to impose connection formulas between subdomains than is practical when analytic results are primarily desired. If, formally, $\lim_{\mu\to 0} U(\mu,\Delta x) = U(\Delta x)$ it is thus reasonable to conjecture that $U(\Delta x)$ provides an approximation to the Euler solution U as well.

An important mathematical difference between the Navier-Stokes equations (2.4) and the Euler equation (6.1) lies in the formulation of boundary conditions. For (2.4) $U(\mu)$ may be prescribed at boundaries as indicated by (2.9) while for (6.1) only certain combinations of U as determined by characteristics are permissible. This reduction of boundary conditions is, of course, a familiar feature of singular perturbation problems.

We now propose to examine how the Euler boundary conditions for $U(\Delta x)$ result from $U(\mu,\Delta x)$ when $\mu \to 0$ when (3.9) is employed.

As described in section 4 the ADI solution method (4.3) used to solve (3.9) can be effectively solved by employing the block-tridiagonal system (4.7) which we now consider in the simplified form

$$-Ja^- U^n_{\ell+1} + b^+ U^n_{\ell-1} + c\, U^n_\ell = g_\ell, \qquad \ell = 1,2,\ldots,L-1, \tag{6.2}$$

where here U^n_0 and JU^n_L are prescribed as inflow and outflow conditions (cf. (2.9)).

With S given by (5.11), let

$$\hat{Q}_x = \begin{pmatrix} 0 & \\ 0 & S[\mathrm{diag}(1,1,\xi_x)]S^{-1}\,\mathrm{sgn}\,u \end{pmatrix} ; \tag{6.3}$$

using (3.3) to define $[C]^{-1}$, (5.17) may be written

$$\lim_{\mu \to 0} Q_x = C\,\hat{Q}_x[C]^{-1}, \tag{6.4}$$

while, according to (4.6),

$$\begin{aligned} \lim_{\mu \to 0} Ja^- &= \tfrac{1}{2}\left[C(\hat{Q}_x - J)[C]^{-1}(I+\lambda_x A)\right], \\ \lim_{\mu \to 0} b^+ &= \tfrac{1}{2}\left[\left((I-J) + C(\hat{Q}_x + J)[C]^{-1}\right)(I+\lambda_x A)\right]. \end{aligned} \tag{6.5}$$

A simple calculation yields

$$(6.6)\qquad S[\mathrm{diag}(1,1,\xi_x)]S^{-1} = (\delta s)^{-1}\begin{pmatrix} \xi_x s_3 - s_2 & 0 & (1-\xi_x)s_2 s_3 \\ 0 & \delta s & 0 \\ \xi_x - 1 & 0 & s_3 - \xi_x s_2 \end{pmatrix},$$

where $\delta s = s_3 - s_2$. Thus, assuming $u > 0$, $\mathrm{sgn}\, u = 1$ so that

$$(6.7)\qquad \hat{Q}_x - J = (\delta s)^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\xi_x - 1)s_3 & 0 & (1-\xi_x)s_2 s_3 \\ 0 & 0 & 0 & 0 \\ 0 & (\xi_x - 1) & 0 & (1-\xi_x)s_2 \end{pmatrix},$$

$$(6.8)\qquad \hat{Q}_x + J = (\delta s)^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\xi_x+1)s_3+2s_2 & 0 & (1-\xi_x)s\ s_3 \\ 0 & 0 & 2\delta s & 0 \\ 0 & (\xi_x-1) & 0 & 2s_3-(\xi_x+1)s_2 \end{pmatrix}.$$

Suppose $\mu \to 0$. For $\ell = L-1$, the coefficient Ja^- in (6.2) determines the influence of the outflow boundary condition JU_L^n; using (6.5) and (6.7) there results:

<u>$M > 1$</u> $(\xi_x = 1)$: here, $\hat{Q}_x - J = 0$.

<u>$M \le 1$</u> $(\xi_x = 0,-1)$: here, $\mathrm{rank}\ (\hat{Q}_x - J) = 1$.

For $\ell = 0$, the coefficient b^+ in (6.2) similarly determines the influence of the inflow boundary condition U_0^n. Now, using (6.7) and (6.8), there results

$\underline{M > 1}$ $(\xi_x = 1)$: $\quad$ $\text{rank}(\hat{Q}_x + J) = 3,$

$\underline{M \le 1}$ $(\xi_x = 0,-1)$: $\quad$ $\text{rank}(\hat{Q}_x + J) = 2,$

i.e., rank $b^+ = 4$ $(M > 1)$, rank $b^+ = 3$ $(M \le 1)$.

Thus, the number of boundary conditions for (3.9) which are effective when $\mu \to 0$ may be summarized as:

	Outflow	Inflow
$M > 1$	0	4
$M \le 1$	1	3

These are exactly the number of boundary conditions which are appropriate for the Euler equations (6.1).

For small values of μ the terms $\kappa_x C$ in which arise in evaluating $a^{\pm}$, $b^{\pm}$ in (4.6) may be retained while using, at the same time, the asymptotic approximation for Q_x given by (6.4). If this approximation is used in (4.3) and if terms proportioned to μ in evaluating the matrices R_x, R_y (defined by (4.2)) in (4.3) are neglected, the result leads, we assert, to a dissipative finite difference scheme for treating (6.1) in which only the hyperbolic boundary conditions which arise from the limit $\mu \to 0$ influence the calculations to any significant extent.

Concluding Remarks

This paper has described a class of compact finite difference equations (3.9) for treating the Navier-Stokes equations when written in the form (2.4). For model problems in which the coefficient matrices appearing in these equations are symmetric and constant the resulting scheme can be shown to be convergent for all values of the mesh parameters $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$ and also to provide second-order accuracy. In this theory the influence of the viscosity μ primarily determines the size of the computational subdomains within which variations in the coefficient matrices A and B can be regarded as small.

An important feature of the finite difference scheme (3.9) is that the natural physical boundary conditions are employed; when $\mu \to 0$ only the boundary conditions for the Euler problem influence the solution.

Acknowledgments

The study of the relationship of the scheme (6.7) to the Euler equations as described in section 6 was partially motivated by the work of MacCormack [2]. Another important motivation were questions raised in personal discussions with R. Beam and R. Warming concerning the related paper of Philips and Rose [3].

References

[1] H. B. Keller, *Numerical solution of two-point boundary value problems*, Regional Conference Series in Applied Mathematics, 24, SIAM, 1976.

[2] R. W. MacCormack, *An efficient explicit-implicit-characteristic method for solving the compressible Navier-Stokes equations*, SIAM-AMS Proceedings, Vol. 11, 1978.

[3] R. Philips and M. E. Rose, *Compact finite difference schemes for mixed initial-boundary value problems*, ICASE Report No. 81-18, June 15, 1981.

[4] J. C. Strikwerda, *Initial boundary value problems for incompletely parabolic systems*, Comm. Pure and Applied Math., Vol. XXX, 1977, pp. 797-822.